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Copenhagen Interpretation of Quantum Mechanics Is Incorrect

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Abstract

(A point-by-point response to a comment (quant-ph/0509130) on our paper (quant-ph/0509089) is added as Appendix C. We find the comment incorrect.)

Einstein's criticism of the Copenhagen interpretation of quantum mechanics is an important part of his legacy. Although most physicists consider Einstein's criticism technically unfounded, we show that the Copenhagen interpretation is actually incorrect, since Born's probability explanation of the wave function is incorrect due to a false assumption on "continuous probabilities" in modern probability theory. "Continuous probability" means a "probability measure" that can take every value in a subinterval of the unit interval $(0, 1)$. We prove that such "continuous probabilities" are invalid. Since Bell's inequality also assumes "continuous probabilities", the result of the experimental test of Bell's inequality is not evidence supporting the Copenhagen interpretation. Although successful applications of quantum mechanics and explanation of quantum phenomena do not necessarily rely on the Copenhagen interpretation, the question asked by Einstein 70 years ago, i.e., whether a complete description of reality exists, still remains open.

Keywords: Foundations of quantum mechanics, Copenhagen interpretation of quantum mechanics, Born's probability explanation of wave function, Bell's inequality

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I. INTRODUCTION

Although Albert Einstein made many fundamental contributions to the development of quantum mechanics, he remained critical to the Copenhagen interpretation of this theory [1]. Niels Bohr was the main defender against Einstein's criticism. Their celebrated debate lasted for more than a decade. Most physicists, however, consider this part of the story of Einstein's life somehow ironic. By simply taking the quantum mechanical description as reality itself, most physicists nowadays have put the issue raised by Einstein, i.e., whether the quantum mechanical description of physical reality is complete [2], behind them. This is largely due to the result of the experimental test of Bell's inequality [3]. However, in contrast to commonly accepted belief, we shall show that the quantum mechanical description (based on the Copenhagen interpretation) is actually incorrect.

Unlike Einstein's criticism, which might be due to his insistence on causality [1], the basis of our claim above is of a technical nature. We prove that Born's probability explanation of the wave function is incorrect (Section II), and show that the experimental result of Bell's inequality is not evidence supporting the Copenhagen interpretation (Section III).

Besides the proof in Section II, Appendix A contains two more involved versions of the proof. Appendix B discusses the hypothetical nature of "continuous probability", which is an incorrect assumption adopted in modern probability theory and causes the falsity of Born's probability explanation.

II. FALSITY OF BORN'S PROBABILITY EXPLANATION

The wave function, denoted by ψ, is the solution of Schrödinger's equation governing a particle. According to Born's explanation, the normalized $|\psi|^2$ is a "probability density function", which implies a "continuous probability", i.e., a "probability measure" whose range includes an interval. However, assuming "continuous probabilities" is a fundamental flaw in modern probability theory. Actually, the range of any probability measure cannot include intervals, so "continuous probabilities" are invalid. In the following, we give a rigorous mathematical proof.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, where Ω is a sample space, $\mathcal{F}$ a collection (σ-algebra) of subsets of Ω, and P the probability measure. We shall not consider any trivial cases, such

as a probability space of a degenerate random variable. For a probability space of a random variable, we assume that the random variable does not take on $\pm\infty$ as its value.

Definition 1 *A P-collection is a nonempty family of sets in $\mathcal{F}$, such that the sets are pairwise disjoint, and each set has a positive probability less than one.*

For a P-collection F, define

$$\Phi(F) = \{\gamma : \ \gamma = P(A), \ A \in F\}$$

and let $G(\mathcal{F})$ be the set of all P-collections in $\mathcal{F}$. Thus, $\cup_{F\in G(\mathcal{F})}\Phi(F)$ is the union of all $\Phi(F)$, taking account of every P-collection in $\mathcal{F}$.

Lemma 1 *The set $\cup_{F\in G(\mathcal{F})}\Phi(F)$ includes all values in $(0,1)$ that the probability measure P can take.*

Proof: Let $\gamma \in (0,1)$ be a value of P. There exists $A \in \mathcal{F}$ with $P(A) = \gamma$. Two complementary sets form a P-collection $F = \{A, A^c\}$, with $\Phi(F) = \{\gamma, 1-\gamma\}$. Therefore, $\gamma \in \cup_{F\in G(\mathcal{F})}\Phi(F)$.

□

Lemma 2 *A P-collection of a probability space $(\Omega, \mathcal{F}, P)$ is countable, i.e., there are countably many (finite or countably infinite) different sets in a P-collection.*

Proof: For a P-collection F of the probability space, we have

$$F = \bigcup_{n=1}^{\infty} H_n$$

where

$$H_n = \{A \in F : 1/(n+1) < P(A) \leq 1/n\}, \ n = 1, 2, \cdots .$$

If F is uncountable, i.e., if there are uncountably many different sets in F, then at least one of $H_1, H_2, \cdots$ must be uncountable. Let H_m, where $1 \leq m < \infty$, be uncountable. Select different sets in H_m, and denote the selected sets by $A_i, i = 1, 2, \cdots$. Clearly, $\{A_i, i = 1, 2, \cdots\}$ is a family of pairwise disjoint sets. Since $P(A_i) > 1/(m+1)$ for all A_i, we have $\sum_{i=1}^{\infty} P(A_i) = \infty$. But this is impossible. So F must be countable.

□

Theorem 1 *For a probability space $(\Omega, \mathcal{F}, P)$, there are values almost everywhere in $(0,1)$ that the probability measure P cannot take.*

Proof: From Lemma 1, we need only show that $\cup_{F\in G(\mathcal{F})}\Phi(F)$ is a nullset, i.e., a set of (Lebesgue) measure zero. From Lemma 2, any P-collection F of the probability space is countable. So any $\Phi(F)$ is countable, and hence is a nullset. From the definition of a nullset, $\Phi(F)$ can be covered by a sequence of open intervals of arbitrarily small total length, i.e., $\Phi(F)$ is a subset of the union of the covering intervals.

On the other hand, there is a countable base $\mathcal{B}$ for the topology induced by the usual metric on the real line (restricted on the interval $(0,1)$). Thus, for any $\Phi(F)$, each covering interval I of $\Phi(F)$ is a union of some members of $\mathcal{B}$. Since the length of I can be arbitrarily small, the measure of any member of $\mathcal{B}$ contained in I can also be arbitrarily small.

Consequently, $\cup_{F\in G(\mathcal{F})}\Phi(F)$ is a subset of a union of the members of $\mathcal{B}$, such that each member is contained in a covering interval of some $\Phi(F)$, and has an arbitrarily small measure. Since any member of $\mathcal{B}$ is a countable union of pairwise disjoint open intervals, $\cup_{F\in G(\mathcal{F})}\Phi(F)$ is covered by a sequence of open intervals of arbitrarily small total length. Therefore, $\cup_{F\in G(\mathcal{F})}\Phi(F)$ is a nullset.

$\square$

We have also prepared two other versions of the proof of Theorem 1. Both versions are essentially the same as the proof given above, but involve more intensive deliberation. Since not every reader would consider the extra versions necessary, we put them in Appendix A.

A "continuous probability" is a "probability measure" whose range includes an interval. We have an immediate consequence of Theorem 1.

Corollary 1 *"Continuous probabilities" are invalid.*

However, modern probability theory assumes "continuous probabilities" in two cases.

Case I. Consider a "continuous random variable". By "continuous" in "continuous random variable", we mean "absolutely continuous", i.e., the "random variable" has a "probability density function", and the "probability measure" of the "random variable" can take every value in a subinterval of $(0,1)$, as exemplified by those with uniform, exponential, and normal distributions. The conclusion below is an immediate corollary of Theorem 1.

Corollary 2 *"Continuous random variables" do not exist.*

Although "continuous random variables" do not exist, there is a popular explanation of "continuous probabilities" based on "continuous random variables": A "continuous random variable" is a result of approximating a sum of n discrete random variables as n tends to infinity.

For example, consider n independent and identically distributed random variables, such that the possible values of each random variable are 0 and 1. As claimed by the De Moivre-Laplace limit theorem, for large n, a function, defined by an integral, can approximate the distribution of the normalized sum of the random variables. The sum represents the normalized number of successes in n Bernoulli trials.

Although the function given by the integral, known as the "standard normal distribution", is considered the limit of the distribution of the normalized sum, the approximation does not result in the so-called "standard normal random variable". For any given n, the set of the possible values of the sum of the n random variables is $\{0, 1, 2, ..., n\}$. Consequently, the normalized sum for any given n has $n + 1$ possible values. No matter how large n is, the possible values of the normalized sum can only form a countable set. In other words, the normalized sum is a discrete random variable for any n.

However, the "normal random variable" is a "continuous random variable" with an uncountable set of values. The approximation is only in the sense that the value of the distribution of the normalized sum and the value of the "normal distribution" can be close. But such approximation does not necessarily imply the closeness between the normalized sum and the "normal random variable" as two functions. The normalized sum has a countable set as its range (set of possible values). But the range of the "normal random variable" is an uncountable set.

Case II. The other way leading to "continuous probabilities" is due to denumerable sequences of elementary "events". Such a sequence corresponds to a decimal expansion of a real number in the unit interval. In general, the base of the expansion can be any given integer $q > 0$. For simplicity and without loss of generality, we let $q = 2$. The following result is also immediate from Theorem 1.

Corollary 3 *Let $\Omega = \{\omega_1, \omega_2, \dots\}$. If $\mathcal{F}$ includes all subsets of Ω, and if $P(\omega_n) = 2^{-n}$ for all $n \geq 1$, then $(\Omega, \mathcal{F}, P)$ is not a valid probability space.*

"Continuous probabilities" are hypothetical, introduced into probability theory through

various assumptions on sample spaces, σ-algebras, and "probability measures", and will not exist if the assumptions are abandoned as they should be, since we have shown that such assumptions lead to contradictions and hence are incorrect. Actually, such assumptions are just different ways to say that a "probability measure" can take every value in a subinterval of $(0, 1)$. Consequently, any "proof" of the existence of "continuous probabilities" (e.g., in case II) is nothing but a tautology. In Appendix B, we show that all "counterexamples" to Theorem 1 are based on such tautology, and clarify a confusion in some arguments of the "counterexamples" caused by misunderstanding on measure theory.

Although "continuous probabilities" are invalid, a discrete probability can be induced by a "continuous probability". For example, consider a "probability density function" f with domain D, which is an interval on the real line. Denote by E a partition of D, i.e., E is a sequence of pairwise disjoint subintervals $E_1, E_2, \cdots$, such that $\cup E_i = D$. By letting each E_i in the partition represent an elementary event "$x \in E_i$" with probability $P(E_i) = \int_{E_i} f(x)dx$, we then obtain a discrete probability.

However, once a partition E is given, for any subinterval H of an elementary event E_i in E, we cannot calculate the probability of $x \in H$, since "$x \in H$" is neither an elementary event, nor deducible from other elementary events. This leads to an uncertainty for the induced probability itself, although any probability is a description of some uncertainty. Such uncertainty in the induced probability is inevitable. This is because assigning a positive probability value to every subinterval of D will lead to a "continuous probability", which is invalid. Nevertheless, with finer and finer partitions, we can decrease the uncertainty. But since different partitions correspond to different probability spaces, the refinement will require infinitely many probability distributions.

Discrete probability functions as induced above are actually used to calculate the values of various probabilities and statistical quantities, not only in quantum mechanics, but also in other applications of probability theory generally. This explains why probability theory works well numerically, although "continuous probabilities" are invalid.

"Continuous probabilities" and the mathematical facts used in this paper to disprove the existence of "continuous probabilities" are well-known, and can be found in standard textbooks. For example, see [4, 5, 6]. Physicists may find [7] more accessible.

III. DISCUSSION AND CONCLUDING REMARKS

The explanation of quantum phenomena and successful applications of quantum mechanics do not necessarily rely on the Copenhagen interpretation. For example, without Born's probability explanation, the solution of Schrödinger's equation is sufficient to show the existence of discrete energy levels.

Most physicists consider the result of the experimental test of Bell's inequality evidence supporting the Copenhagen interpretation. However, the derivation of Bell's inequality assumes "continuous probabilities" [3]. Since "continuous probabilities" are invalid, Bell's inequality itself is incorrect, and hence is not a valid basis for a test. So such result is not supporting evidence for the Copenhagen interpretation. On the other hand, due to the flaw in probability theory, the Copenhagen interpretation is incorrect, and hence is not eligible for a meaningful test.

After pointing out the flaw of the Copenhagen interpretation, we find ourselves in a situation described by Professor Sir Hermann Bondi [8]: Our work might be brushed aside with comments like: "Quantum mechanics works. So there must be some fault in your argument. Why waste time to sort it out when there are so many fascinating things to be done?" However, Einstein would definitely disagree with such comments. Pursuing the truth is not a waste of time in any sense. The Copenhagen interpretation actually closed the door of exploring the reality behind quantum mechanics, though Einstein had tried to keep the door open. With this paper, we want to reopen the door. We conclude by citing Einstein, Podolsky, and Rosen [2]:

"While we have thus shown that the wavefunction does not provide a complete description of reality, we have left open the question of whether or not such a description exists. We believe, however, that such a theory is possible."

APPENDIX A: MORE INVOLVED PROOFS

The following two proofs (Versions A and B) of Theorem 1 are essentially the same as that we have originally given in Section II, but involve more intensive deliberation. For a set S on the real line, $\mu(S)$ is the (Lebesgue) measure of S. If S is an interval, then $\mu(S) = |S|$ is the length of S. To avoid misunderstanding or confusion, we first recall the definition of

a set (on the real line) of (Lebesgue) measure zero.

Definition 2 *A set Z is a set of measure zero, if for each $\epsilon > 0$, there is a sequence of (open) intervals $\{I_m\}$, such that $\cup I_m \supset Z$, and $\sum |I_m| < \epsilon$.*

For convenience of exposition, we refer to the "sequence" in Definition 2 (i.e., in "for each $\epsilon > 0$, there is a sequence ...") as a "sequence of intervals of arbitrarily small total length." From Definition 2, the following is immediately evident.

Any fixed sequence of intervals is not a sequence of intervals of arbitrarily small total length, since the total length of a fixed sequence of intervals cannot be less than each $\epsilon > 0$. Moreover, any fixed sequence of intervals covering a set Z is not relevant to whether Z is of measure zero. For example, let Z be a set on the real line, and $\{I_m\}$ a fixed sequence of intervals of total length l (i.e., $\sum |I_m| = l$), such that $\cup I_m \supset Z$. Although $\sum |I_m| < \epsilon$ does not hold for each $\epsilon > 0$, and although there are surely $0 < \delta < l$ and $I_m \in \{I_m\}$ with $|I_m| \geq \delta$, Z can still be a set of measure zero. Actually, we have the following alternative definition.

Definition 3 *A set Z is a set of measure zero, if for each $0 < \epsilon < l$, where l is arbitrarily given, there is a sequence of (open) intervals $\{I_m\}$, such that $\cup I_m \supset Z$, and $\sum |I_m| < \epsilon$.*

Clearly, Definitions 2 and 3 are equivalent. Let l in Definition 3 be the total length of a fixed sequence of intervals covering a set of measure zero. Since Definition 3 (and hence Definition 2) does not involve the fixed sequence of intervals of total length l, and since l is arbitrary, a set of measure zero is irrelevant to any fixed sequence of intervals.

Version A

Let Z be a set on the real line with $\mu(Z) = 0$, and $\mathcal{I}(Z)$ the family of sequences of intervals covering Z, i.e.,

$$\mathcal{I}(Z) = \{\{I_m\} : \cup I_m \supset Z,\ I_1, I_2, \ldots \text{ are intervals}\}.$$

Lemma 3 *For any decreasing sequence of positive real numbers $\{\epsilon_m\}$ (i.e., $\epsilon_m > \epsilon_{m+1}$ for all $m \geq 1$), there is a sequence $\{I_m\} \in \mathcal{I}(Z)$, such that $|I_m| < \epsilon_m$ for any $I_m \in \{I_m\}$. Clearly, for any subinterval J of I_m, $|J| \leq |I_m| < \epsilon_m$.*

Proof: Assume that the lemma is false. There is then a decreasing sequence of positive real numbers $\{\epsilon_m\}$, such that any $\{I_m\} \in \mathcal{I}(Z)$ contains some I_m with $|I_m| \geq \epsilon_m$, where $\epsilon_m \in \{\epsilon_m\}$. Write

$$M = \sup\{m : |I_m| \geq \epsilon_m, |I_j| < \epsilon_j, j = 1, 2, \ldots, m-1, \{I_m\} \in \mathcal{I}(Z)\}.$$

Since $M = \infty$ implies the existence of $\{I_m\} \in \mathcal{I}(Z)$ with $|I_m| < \epsilon_m$ for all $m \geq 1$, we have $M < \infty$. Thus, for any $\{I_m\} \in \mathcal{I}(Z)$, $\sum |I_m| > \epsilon_M > 0$. As a result,

$$\mu(Z) = \inf\left\{\sum |I_m| : \{I_m\} \in \mathcal{I}(Z)\right\} \geq \epsilon_M > 0.$$

We see a contradiction. Therefore, the lemma is true.

$\square$

From Lemma 3, given P-collection F, for any decreasing sequence of positive real numbers $\{\epsilon_m\}$, we have $\{I_m\} \in \mathcal{I}(\Phi(F))$ with $|I_m| < \epsilon_m$ for any $I_m \in \{I_m\}$. Evidently, ϵ_m can be arbitrarily small, i.e., ϵ_m can be less than any $\epsilon > 0$, for all $m \geq 1$ (e.g., we may let $\epsilon_1 < \epsilon$). On the other hand, Lemma 3 applies in particular if we require every $\{I_m\} \in \mathcal{I}(\Phi(F))$ to be a sequence of open intervals. In this case, for any $\{I_m\} \in \mathcal{I}(\Phi(F))$, $\cup I_m$ equals a countable union of members of the countable base $\mathcal{B}$. Let $B_i(F)$ be the ith member in the union of the members of $\mathcal{B}$ for the P-collection F, and $\mathbf{N}$ the set (or a finite subset) of positive integers. Thus, for each P-collection F, we have $\Phi(F) \subset \cup_{i\in\mathbf{N}} B_i(F)$, and

$$\bigcup_{F\in G(\mathcal{F})} \Phi(F) \subset \bigcup_{F\in G(\mathcal{F})} \bigcup_{i\in\mathbf{N}} B_i(F). \tag{A1}$$

For simplicity and without loss of generality, let each member of $\mathcal{B}$ be an open interval. For example, $\mathcal{B}$ can be the family of open intervals in $(0, 1)$ with rational endpoints. Since $\mathcal{B}$ is a countable base, we can surely write

$$\bigcup_{F\in G(\mathcal{F})} \bigcup_{i\in\mathbf{N}} B_i(F) = \bigcup_{j\in\mathbf{N}} B_j$$

where all $B_j \in \mathcal{B}$.

As shown by Lemma 3, for any P-collection F, we can choose $\{I_m\} \in \mathcal{I}(\Phi(F))$ with $|I_m| < \epsilon_m$, where ϵ_m is sufficiently small for all $m \geq 1$. Since for any $j \geq 1$, B_j is a subinterval of some $I_m \in \{I_m\}$, where $\{I_m\} \in \mathcal{I}(\Phi(F))$ for some P-collection F, we have $|B_j| \leq |I_m| < \epsilon_m$. We can of course let ϵ_m in the above inequality be less than $2^{-j}\tau$ for any

$\tau > 0$. Consequently,

$$\bigcup_{F \in G(\mathcal{F})} \Phi(F) \subset \bigcup_{j \in \mathbf{N}} B_j$$

and

$$\sum_{j \in \mathbf{N}} |B_j| < \sum_{j \in \mathbf{N}} 2^{-j}\tau \leq \tau.$$

The last sum equals τ if $\mathbf{N}$ is the set of all positive integers. Thus, $\cup_{F \in G(\mathcal{F})} \Phi(F)$ is a set of measure zero.

Version B

We begin with (A1) established in Version A. But we no longer require $B_i(F)$ to be intervals. For a P-collection F, write

$$C(F) = \bigcup_{i \in \mathbf{N}} B_i(F)$$

where $B_i(F)$ does not appear in $C(F')$ for any P-collection $F' \neq F$. We list $B_i(F)$ only once in $\cup_{F \in G(\mathcal{F})} \cup_{i \in \mathbf{N}} B_i(F)$. As a result, any $B_i(F)$ appears only in one $C(F)$. Since there are at most countably many $B_i(F)$ in $\cup_{F \in G(\mathcal{F})} \cup_{i \in \mathbf{N}} B_i(F)$. there are at most countably many $C(F)$. So we can use $j = 1, 2, \cdots$ to label different $C(F)$, i.e., for each $C(F)$, there is a unique positive integer j, such that we can denote $C(F)$ by C_j. Consequently,

$$\bigcup_{F \in G(\mathcal{F})} C(F) = \bigcup_{j \in N} C_j$$

and

$$\bigcup_{F \in G(\mathcal{F})} \Phi(F) \subset \bigcup_{j \in \mathbf{N}} C_j.$$

Clearly, each C_j either equals the union of a sequence of open intervals covering $\Phi(F)$ for some P-collection F, or equals a subset of the union of the covering sequence. Since the total length of the covering intervals can be less than any $\epsilon > 0$, the measure of C_j, i.e., $\mu(C_j)$, can be less than any $\epsilon > 0$.

Moreover, since C_j is a nonempty open set, we can express C_j as a countable union of pairwise disjoint open intervals. Denote by $I_{j,k}$ such intervals. So

$$C_j = \bigcup_{k \in \mathbf{N}} I_{j,k}.$$

Since $\mu(C_j)$ can be less than any $\epsilon > 0$, $|I_{j,k}|$ can be less than any $\epsilon > 0$ for any $I_{j,k} \subseteq C_j$. Thus, we can surely let

$$|I_{j,k}| < 2^{-(j+k)}\tau$$

for any $\tau > 0$ (see also Lemma 3). Therefore,

$$\bigcup_{F \in G(\mathcal{F})} \Phi(F) \subset \bigcup_{j \in \mathbf{N}} \bigcup_{k \in \mathbf{N}} I_{j,k}$$

and

$$\sum_{j \in \mathbf{N}} \sum_{k \in \mathbf{N}} |I_{j,k}| < \sum_{j \in \mathbf{N}} \sum_{k \in \mathbf{N}} 2^{-(j+k)}\tau \leq \tau.$$

The last sum equals τ if $\mathbf{N}$ is the set of all positive integers. So $\mu(\cup_{F \in G(\mathcal{F})} \Phi(F)) = 0$.

APPENDIX B: HYPOTHETICAL NATURE OF "CONTINUOUS PROBABILITIES"

One might argue that, with $\Omega, \mathcal{F}$ and P as given in Corollary 3, $(\Omega, \mathcal{F}, P)$ is not only a valid probability space, but also a counterexample to Theorem 1. Based on the same argument, one might even invent various "counterexamples" to Theorem 1. For instance, one might consider the "probability space" of any "continuous random variable" such a counterexample. However, the reasoning behind the above argument does not make any sense, since it is a tautology. Any "continuous probability" is merely an assumption in the disguise of a definition.

A probability measure P is a function, defined on $\mathcal{F}$, a σ-algebra of subsets of a sample space Ω. The set of all values of P is the range of P. Clearly, the range is part of the definition of P.

Therefore, if one claims that the range of a "probability measure" includes an interval, then this property of "continuous probabilities" is actually part of the definition of the "probability measure". Although one may verify this property against the definition, the verification is not a proof of the validity of the "probability measure" itself, since the "probability measure" is just so defined. Any mathematical reasoning, which begins with a definition and ends merely with the definition, is nothing but a tautology.

Some arguments in the "counterexamples" also reflect misunderstanding on measure theory. The following argument is representative. Denote by μ the Lebesgue measure. Let

$I(a,b)$ represent an interval on the real line with endpoints a and b, where $a < b$. Let r be an arbitrary number in $I(a,b)$, and denote by $\{r\}$ the set consisting of only one element r. The values assigned to $I(a,b)$ and $\{r\}$ by the Lebesgue measure are respectively $b-a$ and 0.

The basis of the argument is $I(a,b) = \cup_{r\in I(a,b)}\{r\}$. By letting $a=0$ and $b=1$, one might use $I(0,1) = \cup_{r\in I(0,1)}\{r\}$, together with $\mu(I(0,1)) = 1$ but $m(\{r\}) = 0$ for any $r \in I(0,1)$, to construct a "counterexample" to Theorem 1, and argue that $\mu(\cup_{r\in I(0,1)}\{r\}) = 1$ based on $I(0,1) = \cup_{r\in I(0,1)}\{r\}$. With such argument, one might consider that $\mu(I(0,1))$ is an accumulation of $\mu(\{r\})$ for all $r \in I(0,1)$, i.e., $\mu(I(0,1))$ equals 1 by means of addition of all $\mu(\{r\})$ rather than by definition. This is incorrect.

Consider, in general, a measure space $(I(a,b), \mathcal{F}, \mu)$, where $\mathcal{F}$ is a σ-algebra, and μ is the Lebesgue measure. In measure theory, countable additivity

$$\mu(\cup_{i\in N} A_i) = \sum_{i\in N}\mu(A_i)$$

with pairwise disjoint $A_i \in \mathcal{F}, i = 1, 2, \cdots$ cannot be extended to uncountable additivity, such as $\mu(\cup_{r\in I(a,b)}\{r\}) = \sum_{r\in I(a,b)}\mu(\{r\})$. By definition (e.g., see [6])

$$\sum_{r\in I(a,b)}\mu(\{r\}) = \sup\left\{\sum_{r\in A}\mu(r) : A \subset I(a,b),\ A \text{ is finite}\right\} = 0.$$

The calculated value of $\sum_{r\in I(a,b)}\mu(\{r\})$ contradicts the measure value assigned to the interval $I(a,b)$ according to the definition of Lebesgue measure. Thus, rather than being the result of summation of uncountably many zeros, it is just so defined that $\mu(I(a,b)) = b-a$. Actually, for a measure space, countable union cannot be extended to uncountable union. In particular, we have the following result.

Theorem 2 *The expression $I(a,b) = \cup_{r\in I(a,b)}\{r\}$ is invalid for Lebesgue measure.*

Proof: Any set A with $\mu(A) > 0$ has a non-measurable subset. For example, let $I(a,b) = [0,1)$. A non-measurable subset of $[0,1)$ is given in [6]. If Theorem 2 is false, then $\mu(\cup_{r\in I(a,b)}\{r\}) = \mu(I(a,b)) = b-a > 0$. As a result, there is a non-measurable subset V of $\cup_{r\in I(a,b)}\{r\}$. Define

$$W = \cup_{r\in I(a,b)}\{r\} \setminus V.$$

If $W = \emptyset$, then $I(a,b) = \cup_{r \in I(a,b)}\{r\}$ implies that $I(a,b)$ is non-measurable. This is a contradiction. So we assume $W \neq \emptyset$. By the well-ordering principle, there is a (strict) well ordering $\prec$ for W [6]. Write

$$W_r = \{x \in W : x \prec r\},\ r \in W.$$

Let α be the first element of W. Define

$$E = \{r \in W : \{r\} \cup W_r \text{ is measurable}\}.$$

Since $\{\alpha\} \cup W_\alpha = \{\alpha\} \cup \emptyset = \{\alpha\}$, and since $\{\alpha\}$ is measurable, $\alpha \in E$. There are two cases.

(i) The set W has a last element β, and W_r has a last element $\eta(r)$ for each $r \in W$. If $W_r \subset E$, then for any $x \in W_r$, we have $x \in E$. As a result, $\{x\} \cup W_x$ is measurable for any $x \in W_r$. In particular, $\{\eta(r)\} \cup W_{\eta(r)}$ is measurable. So

$$\{r\} \cup W_r = \{r\} \cup (\{\eta(r)\} \cup W_{\eta(r)})$$

is measurable. Consequently, $r \in E$. By induction on W, we have $W = E$. Therefore, $\beta \in E$, and hence

$$\{\beta\} \cup W_\beta = W$$

is measurable. As a result

$$\cup_{r \in I(a,b)}\{r\} \setminus W = V$$

is measurable. We see a contradiction again.

(ii) The set W, or W_r for some $r \in W$, does not have a last element. Define $Z = \{z + d : z \in C\}$, where d is a constant, such that $I(a,b) \cap Z = \emptyset$, and C is the Cantor set. Since $\mu(C) = 0$, and since Lebesgue measure is translation invariant [6], $\mu(Z) = \mu(C) = 0$.

If W_r does not have a last element for some $r \in W$, then we take an element of Z that has not been taken for any $W_s, s \neq r$. Denote this element of Z also by $\eta(r)$, and the set of all such $\eta(r)$ by $\mathcal{H}$. Since $\mathcal{H} \subset Z$, we have $\mu(\mathcal{H}) = 0$. If W_r has a last element for each $r \in W$, then $\mathcal{H} = \emptyset$.

We extend the order $\prec$ by setting $\eta(r) \prec r$ and $x \prec \eta(r)$ for all $x \in W_r$. If W_s and W_r do not have last elements, where $s \prec r$, then with such extension, we have (a) $\eta(s) \prec \eta(r)$,

(b) for any $x \in W_r$, if $x \prec s$, then $x \prec \eta(s)$; otherwise $\eta(s) \prec x$, and (c) $x \prec \eta(r)$ for any $x \in W_s$. Define $U_r = \{\eta(r)\} \cup W_r \cup \{\eta(s) \in \mathcal{H} : s \prec r\}$. For W_r with a last element, $U_r = W_r \cup \{\eta(s) \in \mathcal{H} : s \prec r\}$.

If W does not have a last element, then we take an element not in $\cup_{r\in I(a,b)}\{r\} \cup \mathcal{H}$, denote this element also by β, and define $U = \{\beta\} \cup W \cup \mathcal{H}$. We further extend the order $\prec$ by setting $r \prec \beta$ for all $r \in W \cup \mathcal{H}$. If W already has a last element β, then $\beta \in W$, and $U = W \cup \mathcal{H}$. It is easy to verify that U is well ordered by the extended order $\prec$, and has the same first element as that of W.

Now U has a last element β, and U_r has a last element for each $r \in U$, which is either the last element of W_r, or $\eta(r) \in \mathcal{H}$. We use U and U_r to replace W and W_r, respectively, and consider measure space $(I(a,b) \cup \mathcal{H} \cup \{\beta\}, \mathcal{F} \cup \mathcal{F}(\mathcal{H},\beta), \mu)$ instead of $(I(a,b), \mathcal{F}, \mu)$, where $\mathcal{F}(\mathcal{H},\beta)$ is the family of all subsets of $\mathcal{H} \cup \{\beta\}$. With the same argument for case (i),

$$\{\beta\} \cup U_\beta = U = \begin{cases} \{\beta\} \cup W \cup \mathcal{H}, & \beta \notin W \\ W \cup \mathcal{H}, & \beta \in W \end{cases}$$

is measurable. Consequently,

$$W = \begin{cases} U \setminus (\{\beta\} \cup \mathcal{H}), & \beta \notin W \\ U \setminus \mathcal{H}, & \beta \in W \end{cases}$$

is measurable. This again leads to the contradiction in case (i).

□

We can also obtain the above result based on a simple observation. One of the most important notions in measure theory is that of neglecting sets of measure zero. After neglecting sets of measure zero from the measure space, it can be seen clearly that $I(a,b) = \cup_{r\in I(a,b)}\{r\}$ is invalid.

A probability measure is also a measure. The elucidation above shows clearly why a probability measure must be defined on a σ-algebra $\mathcal{F}$ for an uncountable sample space like $I(a,b)$. In other words, for such a sample space $\Omega = I(a,b)$, we must assign probabilities to subsets of Ω.

For measure spaces like $(I(a,b), \mathcal{F}, \mu)$, a necessary condition is that $\mathcal{F}$ cannot include all subsets of $I(a,b)$. This is because some subsets are not measurable in the sense of Lebesgue measure. So the definition of $\mathcal{F}$ imposes some restrictions on the members of $\mathcal{F}$. Yet

Theorem 1 shows that such restrictions are not restrictive enough to make $(I(a,b), \mathcal{F}, P)$, where $P = \mu/(b-a)$, a valid probability space. A more stringent restriction that the range of the probability measure must be a set of Lebesgue measure zero should be imposed on any probability space. With this restriction, the σ-algebra $\mathcal{F}$ of the probability space $(I(a,b), \mathcal{F}, P)$ cannot include all subintervals of $I(a,b)$.

APPENDIX C: RESPONSE TO COMMENT (QUANT-PH/0509130)

This appendix is a point-by-point response to a comment (quant-ph/0509130) on our paper (quant-ph/0509089). We find the comment incorrect.

1. Points Raised in quant-ph/0509130

Point 1: The comment (quant-ph/0509130) claims a "counterexample" $(\Omega, \mathcal{F}, P)$, where $\Omega = [0,1], \mathcal{F} =$ Borel sets in $[0,1]$, and P is the Lebesgue measure (referred to as Lebesgue-Borel measure in quant-ph/0509130) restricted to the σ-algebra $\mathcal{F}$ of the Borel sets. With this "counterexample", it is claimed, in quant-ph/0509130, that $P(\mathcal{F})$ (the range of P) is $[0,1]$.

Point 2: Let $S = \cup_{F \in G(\mathcal{F})} \Phi(F)$ (the right side is generally defined in our paper quant-ph/0509089). It is claimed, in quant-ph/0509130, that S is not necessarily a member of the σ-algebra $\mathcal{F}$.

Point 3: From $P(\Phi(F)) = 0$ for any $F \in G(\mathcal{F})$, one cannot conclude $P(S) = \sum_{F \in G(\mathcal{F})} P(\Phi(F)) = 0$.

2. Our Response

Response to Point 1: The "counterexample" is a meaningless tautology. We have intensively deliberated on this issue in our paper. Please see Appendix B in quant-ph/0509089.

Any "continuous probability", like P given in the "counterexample", is merely an assumption in the disguise of a definition. The range of P is part of the definition of P. Such definition causes contradictions, as we have proved in quant-ph/0509089 that the range of a probability measure cannot include any interval. The definition of P in the "counterexam-

ple" also makes the comment in quant-ph/0509130 self-contradictory.

To see the self-contradiction, here is an example. If P is a probability measure, then any element of S is a value of P in $(0, 1)$. Denote by $\mathcal{R}$ the range of P after removing 0 and 1. So $\mathcal{R}$ is a subset of S, and $S \subset (0, 1)$. On the other hand, if the range of P is $[0, 1]$ as claimed in quant-ph/0509130 (Point 1), then $\mathcal{R} = (0, 1)$ and $S = (0, 1)$. Clearly, the open unit interval $(0, 1)$ is a member of the σ-algebra $\mathcal{F}$ of the Borel sets in $[0, 1]$. This contradicts the claim in Point 2 that S is not necessarily a member of $\mathcal{F}$.

Response to Point 2: We have proved that, for a probability measure, $\cup_{F\in G(\mathcal{F})}\Phi(F)$ is a set of Lebesgue measure zero in quant-ph/0509089. Since Lebesgue measure restricted to the σ-algebra of Borel sets is not complete, a set of Lebesgue measure zero is not necessarily a member of the σ-algebra.

Response to Point 3: Point 3 is misleading, since our proof of the invalidity of "continuous probabilities" in quant-ph/0509089 is based on a concept in topology called countable base, and does not involve $P(S) = \sum_{F\in G(\mathcal{F})} P(\Phi(F)) = 0$. We give a concise version of the proof in the main text, and two additional versions with more intensive deliberations in Appendix A. Point 3 is irrelevant to any of the three versions of our proof.

3. Conclusion

We have responded, point-by-point, to the comment in quant-ph/0509130 on our paper quant-ph/0509089. We conclude that the comment is incorrect.

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